



# On uniqueness of decomposition of 4-polyhedron into Cartesian product of the 2-dimensional factors <sup>☆</sup>

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Accepted 10 February 2004

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## Abstract

If decomposition of 4-dimensional polyhedron into Cartesian product of 2-dimensional factors is not unique with respect to homeomorphism, then one of the factors is the same in all decompositions and it is homeomorphic to a bundle of intervals over a graph.

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MSC: 54B10

Keywords: Cartesian product; Polyhedron; Splitting Theorem

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## 1. Introduction

A decomposition of a polyhedron into Cartesian product of 1-dimensional factors is unique [2]. We have considered the uniqueness of decomposition into Cartesian product of 3-dimensional polyhedron in an earlier paper [11]. If the uniqueness of decomposition does not exist, then 1-dimensional factor is an arc. If we consider the decomposition of 4-polyhedron into 3- and 1-dimensional factors, then 3-dimensional factor is not necessarily a polyhedron. Consider a cube  $I^3$  and a wild arc  $\alpha$  in its interior. Let  $A$  be a quotient space  $I^3/\alpha$ . Then  $I^3$  and  $A$  are not homeomorphic but the spaces  $I^3 \times S^1$  and  $A \times S^1$  are homeomorphic. The proof of this fact is similar to the proof of Andrews and Curtis [1] of the fact that  $R^3/\alpha \times R$  is homeomorphic to  $R^4$ . There exist also non-homeomorphic Seifert 3-manifolds with infinite  $\pi_1(M^3), \pi_1(N^3)$  such that  $M^3 \times S^1 \approx N^3 \times S^1$  (see [4, Theorem 11.5]). In these cases 1-dimensional factors are not arcs. 2-dimensional divisors

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<sup>☆</sup> Supported by UG grant BW 5100-5-0319-3.

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of polyhedron are polyhedra by Kosiński's result [8], so if we consider decomposition of 4-polyhedron into 2-dimensional factors both of them are polyhedra. We have described the case when one of the factors is a Cartesian product of 1-dimensional polyhedra in [14]. One-dimensional factors are the same in all decompositions and if a decomposition is not unique, then at least one of 1-dimensional factors is an arc.

If both factors are the same 2-polyhedra, then the decomposition is unique [13]. The question on the Cartesian roots was posed by Ulam in 1933 [15]. If the Cartesian squares of 3-manifolds are homeomorphic, then the factors can be topologically different. In the paper [9] Kwasik and Schultz present the lens spaces which are the counterexamples.

In 1945 Borsuk [3] showed that an  $n$ -dimensional closed and connected manifold without boundary has not more than one decomposition into Cartesian product of topologically prime factors of dimension  $\leq 2$ . This theorem is not true for manifolds with boundaries. The torus with one hole and the disk with two holes are not homeomorphic but their Cartesian products with an interval are homeomorphic. Similarly, the product of the Möbius band with a hole and an interval is homeomorphic to the product of the Klein bottle with a hole and an interval. We can construct all 2-manifolds in above examples by identification of two pairs of disjoint arcs in the boundary of a disk. After Cartesian multiplication with an interval, the order of identified arcs on the boundaries of disks is not essential.

Malešič, Repovš, Rosicki, Zastrow proved the following Theorem, which is essential case in our problem [10].

**Theorem 1.1.** *If  $X, Y, X', Y'$  are compact 2-manifolds with boundaries and the Cartesian products  $X \times Y$  and  $X' \times Y'$  are homeomorphic, then either  $X$  and  $Y$  are homeomorphic to  $X'$  and  $Y'$  (with respect to permutation of factors) or if  $X$  is not homeomorphic to  $X'$  or  $Y'$ , then  $Y$  is homeomorphic to  $I^2$  or to  $S^1 \times I$  and  $Y$  is homeomorphic to  $X'$  or  $Y'$ .*

We used Splitting Theorem (see [6, p. 154] or [7]) to prove the above Theorem. We investigated the boundaries of the manifolds  $X \times Y$  and  $X' \times Y'$ .

**Theorem 1.2** (Splitting Theorem). *Let  $M$  be a compact, sufficiently-large, irreducible, boundary-irreducible 3-manifold. Then there exists a two side, incompressible 2-manifold  $W \subset M$ , unique up to ambient isotopy, having the following three properties:*

- (a) *The components of  $W$  are annuli and tori, and none of them is boundary-parallel in  $M$ .*
- (b) *Each component of  $(\sigma_W(M), \sigma_{\partial W}(\partial M))$  is either a Seifert pair or a simple pair.*
- (c)  *$W$  is minimal with respect to inclusion among all two-sided 2-manifolds in  $M$  having properties (a) and (b).*

In the book [6] manifolds are orientable, so we must assume that the manifold  $M$  is orientable.

We denote by  $\sigma_W(M)$  the 3-manifold obtained by splitting  $M$  along  $W$ . Similarly, we define 2-manifold  $\sigma_{\partial W}(\partial M)$ , which may be naturally identified with a submanifold of the boundary of  $\sigma_W(M)$ . In the case when  $X$  or  $Y$  is non-orientable we worked with their orientable double covers.

In our paper we consider Cartesian products of 2-polyhedra, so we will present the next example. Let us consider a square  $[-1, 1] \times [-1, 1]$ . Let us attach a different graph to each corner of this square. We obtain a 2-polyhedron  $K$ . If we substitute the graph attached to the point  $(1, 1)$  by the graph attached to the point  $(1, -1)$  and the graph attached to the point  $(1, -1)$  by the graph attached to the point  $(1, 1)$  we obtain the second 2-polyhedron  $K'$ , not homeomorphic to the first. Their Cartesian products with an interval  $K \times [-1, 1]$  and  $K' \times [-1, 1]$  are homeomorphic. We can imagine a homeomorphism as a “rotation around the axis  $y = 0, z = 0$ ” by the angle  $\pi$  of the set  $[0.5, 1] \times [-1, 1] \times [-1, 1]$  with fixed the set  $[-1, -0.5] \times [-1, 1] \times [-1, 1]$ .

Of course, the products  $K \times [-1, 1] \times [-1, 1]$  and  $K' \times [-1, 1] \times [-1, 1]$  are homeomorphic too. We consider the previous homeomorphism on two first factors and an identity on the last factor. We can present a Möbius band  $M$  as a quotient space of a square  $[-1, 1] \times [-1, 1]$  with a relation  $(-1, t) \sim (1, -t)$ . It is easy to see that we can obtain a homeomorphism between  $K \times M$  and  $K' \times M$ .

We are going to prove:

**Theorem 1.3.** *If a decomposition of compact, connected 4-polyhedron into Cartesian product of 2-polyhedra is not unique, then in all different decompositions one of the factors is homeomorphic to the same bundle of intervals over a graph.*

## 2. Factors with distinguished points

In this section we discuss the case when one of the factors  $X$  is a polyhedron with distinguished isolated points. We will consider the cases when  $X$  is a connected polyhedron with not empty 1- and 2-dimensional parts, when  $X$  has isolated local cut points, when the set  $n_2 X$ , defined below as in [13], is not empty and when the above conditions do not hold but there exists a non-Euclidean point  $x \in nX$  with the regular neighborhood different from Cartesian product of an interval with a cone  $\{1, \dots, m\}$ . If 2-polyhedron does not have such distinguished points we say that it is *without distinguished points*.

First, as in [13] we define some subsets of non-Euclidean points of a polyhedron  $P$ .

**Definition 2.1.** If  $P$  is  $k$ -dimensional polyhedron, then we define inductively the sets  $n_i P$  for  $i = 1, \dots, k$ .

- (i)  $n_0 P = P$ ;
- (ii)  $n_i P$  denotes the subset of  $n_{i-1} P$  consisting of the points which have no neighborhood homeomorphic to  $R^{k-i+1}$  or to  $R_+^{k-i+1}$  in the set  $n_{i-1} P$ ;
- (iii) We denote the set  $n_1 P$  by  $nP$ .

In the paper [14] we proved Lemma 1, which we present here as:

**Lemma 2.1.** *If  $K = X_1 \times \dots \times X_k$ , where  $X_i$  are polyhedra of dimension at most 2 for  $i = 1, \dots, k$ , then*

$$n_i K = \bigcup \{n_{i_1} X_1 \times \dots \times n_{i_k} X_k : i_p = 0, 1, 2, i_1 + \dots + i_k = i\}.$$

In the special case we obtain:

**Corollary 2.1.** *If  $X$  and  $Y$  are 2-polyhedra, then*

$$n_i(X \times Y) = \bigcup \{n_p X \times n_q Y : p + q = i; p, q \in \{0, 1, 2\}\}.$$

We also will use lemma similar to Lemma 3.2 from [13] or to Lemma 2.1 from [12].

**Lemma 2.2.** *Suppose  $DX, DY, DX'$  and  $DY'$  are nowhere dense subpolyhedra of compact, connected 2-polyhedra  $X, Y, X'$  and  $Y'$ , respectively, and  $F : X \times Y \rightarrow X' \times Y'$  is a homeomorphism such that*

- (i)  $F((X \times DY) \cup (DX \times Y)) = (X' \times DY') \cup (DX' \times Y')$ ;
- (ii)  $F(DX \times DY) = DX' \times DY'$ .

*Then  $F(X \times DY) = X' \times DY'$  or  $F(X \times DY) = DX' \times Y'$ .*

In Lemma 3.2 from [13] we have  $X = Y$  and  $X' = Y'$  but the proof is almost the same as a proof of this lemma.

Using Lemma 2.2 we can easily prove Lemma 2.3, similar to Lemma 3.3 [13].

**Lemma 2.3.** *Suppose  $X, Y, X'$  and  $Y'$  are compact, connected 2-polyhedra,  $F : X \times Y \rightarrow X' \times Y'$  is a homeomorphism and  $Y$  has local cut points. Then  $X$  is homeomorphic to  $X'$  or to  $Y'$  and the second of the polyhedra  $X', Y'$  has the local cut points.*

**Proof.** First we consider the case where there exists a point  $x$  of  $Y$  such that  $\dim_x Y = 1$ . We denote  $DY = \{x \in Y : \dim_x Y = 2\} \cap \text{cl}\{x \in Y : \dim_x Y = 1\}$ . Similarly,  $DX, DX', DY'$ . All these sets are finite (or empty), so they are nowhere dense. The assumptions of Lemma 2.2 hold so assertion is true.

If  $\dim_x X = 2$  for every  $x \in X$  and  $\dim_y Y = 2$  for every  $y \in Y$  then the sets of local cut points  $DX$  and  $DY$  are finite, so they are nowhere dense. The assumptions of Lemma 2.2 hold. So we obtain the thesis.

If there exists a point  $x$  of  $X$  such that  $\dim_x X = 1$  and  $\dim_y Y = 2$  for every  $y \in Y$  we define  $DX$  as in the first case and  $DY$  as in the second and we use Lemma 2.2.

We say that a polyhedron is *prime* if it is not a non-trivial Cartesian product of polyhedra. In [14] we have proved

**Lemma 2.4.** *Let  $K = X_1 \times \cdots \times X_k$  and  $L = Y_1 \times \cdots \times Y_n$ , where  $X_i, Y_i$  are prime polyhedra of dimension at most 2. If  $F : K \rightarrow L$  is a homeomorphism and  $i_p = 0, 1, 2$  for  $p = 1, \dots, k$ , then  $F(n_{i_1} X_1 \times \cdots \times n_{i_k} X_k) = n_{j_1} Y_1 \times \cdots \times n_{j_n} Y_n$  for a system  $(j_1, \dots, j_n)$  of numbers such that  $j_p = 0, 1, 2$  for  $p = 1, \dots, n$  and  $i_1 + \cdots + i_k = j_1 + \cdots + j_n$ .*

Next, again we present lemma similar to Lemma 3.4 from [13], which is a simple consequence of the previous lemma.

**Lemma 2.5.** *Suppose  $X, Y, X'$  and  $Y'$  are compact, connected and prime 2-polyhedra,  $F: X \times Y \rightarrow X' \times Y'$  is a homeomorphism and  $n_2Y \neq \emptyset$ . Then  $X$  is homeomorphic to  $X'$  or to  $Y'$ . If  $X$  is homeomorphic to  $X'$ , then  $n_2Y' \neq \emptyset$ .*

*Moreover,  $F(nX \times nY) = nX' \times nY'$ .*

**Proof.** We have  $F(nX \times nY) = X' \times n_2Y'$  or  $F(nX \times nY) = nX' \times nY'$  or  $F(nX \times nY) = n_2X' \times Y'$  by Lemma 2.4. The first case implies that  $X'$  is not prime, the third case implies that  $Y'$  is not prime, so we have the second case. Then  $F(X \times n_2Y) = X' \times n_2Y'$  or  $F(X \times n_2Y) = n_2X' \times Y'$ , again by Lemma 2.4. If the first case holds, then  $X$  is homeomorphic to  $X'$ , if the second case holds then  $X$  is homeomorphic to  $Y'$ .

If  $X$  and  $Y$  are both not prime, then also  $X'$  and  $Y'$  are both not prime by [14, Theorem 1]. The decomposition into 1-dimensional polyhedra is unique. If only one  $X$  or  $Y$  is not prime, it is easy to see that one of the polyhedra  $X'$  or  $Y'$  is not prime. This case was considered in [14, Theorem 2]. Then 1-dimensional factors are homeomorphic and if 2-dimensional factors are not homeomorphic, then at least one of 1-dimensional factors is an arc.

We need one more lemma.

**Lemma 2.6.** *Suppose  $X, Y, X'$  and  $Y'$  are compact connected 2-polyhedra,  $F: X \times Y \rightarrow X' \times Y'$  is a homeomorphism and there exist points  $x \in nY$  such that there do not exist  $m \in N$  and a closed neighborhood of  $x$  in  $Y$  homeomorphic to  $[0, 1] \times \text{cone}\{1, \dots, m\}$ .*

*Then  $X$  is homeomorphic to  $X'$  or to  $Y'$ . The second of the polyhedra  $X', Y'$  has a point  $x$  such that do not exist  $m \in N$  and a closed neighborhood of  $x$  homeomorphic to  $[0, 1] \times \text{cone}\{1, \dots, m\}$ .*

**Proof.** We can assume that  $Y$  does not have local cut points and  $n_2Y = \emptyset$ . The set of the points of  $nY$  such as in assumption of lemma is finite, so it is nowhere dense. So we denote it by  $DY$ . If  $X$  satisfies the same conditions and we denote analogous subsets of  $X, X'$  and  $Y'$  by  $DX, DX'$  and  $DY'$ , we can easily see that the assumptions of Lemma 2.2 hold, so we use this lemma.

If  $X$  does not satisfy the above conditions but  $X$  does not have such points  $x$  that  $\dim_x X = 1$ , we can join the set  $n_2X$  and the local cut points of  $X$  to the set  $DX$ . The set  $DX$  still is finite and assumptions of Lemma 2.2 still hold.

If  $X$  has locally 1-dimensional part and  $y_0 \in DY$  then it is easy to see that for each component  $A$  of the set  $\{x \in X: \dim_x X = 1\}$  the homeomorphism  $F$  maps the set  $A \times \{y_0\}$  onto set  $A' \times \{y'_0\}$ , where  $A'$  is a component of the set  $\{x' \in X': \dim_{x'} X' = 1\}$ ,  $F(A \times Y) = A' \times Y'$  and  $y'_0 \in DY'$ .

Because, by Lemma 2.2, for each component  $B$  of the set  $\{x \in X: \dim_x X = 2\}$  we have  $F(B \times \{y_0\}) = B' \times \{y'_0\}$ , where  $B'$  is a component of the set  $\{x' \in X': \dim_{x'} X' = 2\}$  and the set  $X \times \{y_0\}$  is connected, then  $F(X \times \{y_0\}) = X' \times \{y'_0\}$ .

So  $X$  and  $X'$  are homeomorphic.

### 3. The set $nY$ contains a simple closed curve

**Lemma 3.1.** *Suppose  $X, Y, X'$  and  $Y'$  are compact connected 2-polyhedra,  $F: X \times Y \rightarrow X' \times Y'$  is a homeomorphism and  $nY$  contains a simple closed curve. Then  $X$  is homeomorphic to  $X'$  or to  $Y'$ .*

**Proof.** We can assume that  $Y$  does not have local cut points and  $n_2Y = \emptyset$  and  $X$  and  $Y$  are prime. Then a simple closed curve  $S$  is a component of  $nY$ . If  $n_2X = \emptyset$ , then by Lemma 2.1 we have  $F((X \times nY) \cup (nX \times Y)) = (X' \times nY') \cup (nX' \times Y')$  and  $F(nX \times nY) = nX' \times nY'$ . If  $n_2X \neq \emptyset$ , then by Lemma 2.5 also  $F(nX \times nY) = nX' \times nY'$ . If  $\dim_x X = 2$  for each  $x \in X$ , then the sets  $nX, nY, nX', nY'$  are nowhere dense in  $X, Y, X', Y'$ , respectively. So by Lemma 2.2 the set  $X \times nY$  is homeomorphic to  $X' \times nY'$  or to  $nX' \times Y'$ . Therefore,  $X \times S^1$  is homeomorphic to  $X' \times S^1$  or to  $S^1 \times Y'$ . Hence,  $X$  is homeomorphic to  $X'$  or  $Y'$  (see [13]).

If the polyhedron  $X$  has a non-empty 1-dimensional part, then only one polyhedron, say  $X'$ , has an analogous non-empty part. Then for each component  $A$  of the set  $\{x \in X: \dim_x X = 1\}$  we have  $F(A \times nY) = A' \times nY'$ , where  $A'$  is a component of  $\{x' \in X': \dim_{x'} X' = 1\}$ . Because our polyhedra are connected, it is easy to see that in this case also  $X \times S^1$  is homeomorphic to  $X' \times S^1$ .

Hence,  $X$  is homeomorphic to  $X'$  (see [13]).

We say that 2-polyhedron is *simple* if it does not contain distinguished points and its non-Euclidean part does not contain a simple closed curve.

### 4. The remaining case

For each component  $A$  of the set  $X \setminus nX$  we can define the 2-manifold with the boundary  $M(A)$ . We can define  $M(A)$  as the set  $A \setminus U(nX)$ , where  $U(nX)$  denotes open regular neighborhood of  $nX$ , but like in [13], for technical reasons, in the sequel we shall use another definition. Except for the isolated points of  $nX$  with the Euclidean neighborhood in  $A$  both definitions are equivalent.

The collection of components of space  $P$  will be denoted by  $\square P$ .

#### Definition 4.1.

- (1) We denote by  $N(A)$  the set of all sequences  $\{x_n\}$  in  $A$  which are convergent in  $X$  and are such that for every neighborhood  $U$  of the point  $\lim x_n$  in  $X$  there exists  $U_0 \in \square(U \setminus nX)$  and a natural number  $n_0$  such that for every  $n > n_0$  we have  $x_n \in U_0$ .
- (2) In the set  $N(A)$  we define the equivalence relation “ $\sim$ ”.  
We have  $\{x_n\} \sim \{y_n\}$  iff
  - (i)  $\lim x_n = \lim y_n = x_0$  in  $X$ ;
  - (ii) for every neighborhood  $U$  of  $x_0$  in  $X$  there exist  $U_0 \in \square(U \setminus nX)$  and a natural number  $n_0$  such that for every  $n > n_0$  we have  $x_n \in U_0$  and  $y_n \in U_0$ .
- (3) By  $M(A)$  we denote the set  $N(A)/\sim$ .

- (4) We define a basis for the topology of  $M(A)$ . Let  $[\{x_n^0\}] \in M(A)$  and  $\lim x_n^0 = x^0$ . Let  $U$  be a neighborhood of the point  $x^0$  in  $X$  and let  $U_0$  denote the component of the set  $U \setminus nX$  such that for almost all  $n$  we have  $x_n^0 \in U_0$ . We denote by  $V(U, [\{x_n^0\}])$  the set of  $[\{x_n\}] \in M(A)$  such that  $\lim x_n \in U$  and  $x_n \in U_0$  for almost all  $n$ . The collection of the sets  $V(U, [\{x_n^0\}])$  is a basis for the topology of  $M(A)$ .

The first definition is simpler than the second, but if we use the second definition, then the following properties (the same as in [13]) are very simple.

**Property 4.1.** *If  $\lim x_n = x$  in  $A$ , then  $[\{x_n\}] = [\{x\}]$  (where  $\{x\}$  is the constant sequence).*

**Property 4.2.** *The function  $h_A: A \rightarrow M(A)$  given by the formula  $h_A(x) = [\{x\}]$  is a topological embedding. Let  $g_A: M(A) \rightarrow \bar{A}$  be given by the formula  $g_A([\{x_n\}]) = \lim x_n$ , then  $g_A \circ h_A$  is an embedding.*

**Property 4.3.** *Let  $X, Y, X'$  and  $Y'$  be 2-polyhedra and let  $F: X \times Y \rightarrow X' \times Y'$  be a homeomorphism. Let  $F(A \times B) = A' \times B'$ , where  $A \in \square(X \setminus nX)$ ,  $A' \in \square(X' \setminus nX')$ ,  $B \in \square(Y \setminus nY)$ ,  $B' \in \square(Y' \setminus nY')$ . There exists a homeomorphism  $F_{A,B}: M(A) \times M(B) \rightarrow M(A') \times M(B')$  such that  $(g'_A \times g'_B) \circ F_{A,B} = F|_{\bar{A} \times \bar{B}} \circ (g_A \times g_B)$ .*

**Remark 4.1.** Denote by  $P_1: A' \times B' \rightarrow A'$  and  $P_2: A' \times B' \rightarrow B'$  the projections on the first and the second factor, respectively. The homeomorphism  $F_{A,B}$  is given by the formula:

$$F_{A,B}([\{x_n\}], [\{y_n\}]) = ([\{P_1 F(x_n, y_n)\}], [\{P_2 F(x_n, y_n)\}]).$$

In this section we can assume that  $Y$  is simple, which means that  $Y$  is a polyhedron without local cut points, such that  $nY$  consists of the disjoint union of arcs and the regular neighborhood of each point  $x \in nY$  is of the form  $\text{cone}\{1, 2, \dots, m\} \times I$ . The same for  $Y'$ . In the opposite case, the polyhedra  $X$  and  $X'$  are homeomorphic. We consider arbitrarily 2-polyhedra  $X$  and  $X'$ . By Lemma 2.4, if  $F: X \times Y \rightarrow X' \times Y'$  is a homeomorphism, then  $F(nX \times Y)$  is equal to  $nX' \times Y'$  or  $X' \times nY'$ . Here we consider the first case, so by [11]  $nX$  and  $nX'$  are homeomorphic. Similarly, 1-dimensional parts of  $X$  and  $X'$  are homeomorphic.

We will prove the following

**Lemma 4.1.** *If 2-polyhedron  $Y$  is not a Cartesian product of an arc and a 1-polyhedron,  $F: X \times Y \rightarrow X' \times Y'$  is a homeomorphism such that  $F(nX \times Y) = nX' \times Y'$ ,  $nX \neq \emptyset$  and  $A \in \square(X \setminus nX)$ ,  $A' \in \square(X' \setminus nX')$  such that  $F(A \times Y) = A' \times Y'$ , then  $M(A)$  and  $M(A')$  are homeomorphic.*

**Proof.** If our polyhedron  $Y$  is not simple, then  $X$  and  $X'$  are homeomorphic.

If for one of components  $B_0 \in \square(Y \setminus nY)$  the surface  $M(B_0)$  is not homeomorphic to a disk  $I^2$  or an annulus  $S^1 \times I$ , then we consider the homeomorphism  $F_{A_0, B_0}: M(A_0) \times M(B_0) \rightarrow M(A'_0) \times M(B'_0)$ . By Theorem 1.1  $M(A_0)$  is homeomorphic to  $M(A'_0)$  or  $M(B'_0)$ . We will prove that  $M(A_0)$  is homeomorphic to  $M(A'_0)$ .

If  $X$  is not simple, then  $M(B_0)$  is homeomorphic to  $M(B'_0)$  because  $Y$  is homeomorphic to  $Y'$ . Therefore,  $M(A_0)$  is homeomorphic to  $M(A'_0)$ .

Let  $X$  be simple and  $nX$  be a disjoint union of arcs. Then  $M(B_0) \times I$  is homeomorphic to  $M(B'_0) \times I$ , so if  $M(A_0)$  is homeomorphic to a disk or annulus, then  $M(A_0)$  is homeomorphic to  $M(A'_0)$ .

Let us assume that  $M(A_0)$  is prime and  $M(A_0)$  is homeomorphic to  $M(B'_0)$ . Then  $M(B_0)$  is homeomorphic to  $M(A'_0)$ .

Now, let us consider the case when there exist more than one component  $A$  of the set  $X \setminus nX$ . If for any of them  $M(A)$  is homeomorphic to  $M(A')$ , then  $M(B_0)$  is homeomorphic to  $M(B'_0)$  and therefore  $M(B_0)$  is homeomorphic to  $M(A_0)$ . In the opposite case all  $M(A)$  are homeomorphic to  $M(B'_0)$  and all  $M(A')$  are homeomorphic to  $M(B_0)$ . Let us consider such a component  $A$  that  $\bar{A}_0 \cap \bar{A} \neq \emptyset$  and two arcs  $J_0 \subset \partial M(A_0)$ ,  $J \subset \partial M(A)$  such that they are components of preimages of a component of  $nX \cap \bar{A}_0$  by  $g_{A_0}$  and  $nX \cap \bar{A}$  by  $g_A$ . We obtain new surface  $M$  gluing  $J_0$  and  $J$  by  $g_{A_0}|_{J_0}$  and  $g_A|_J$ . We construct an analogous surface  $M'$  by analogous glueing of surfaces  $M(A'_0)$  and  $M(A')$ . The Cartesian products  $M \times M(B_0)$  and  $M' \times M(B'_0)$  are homeomorphic. If  $M(B'_0)$  is homeomorphic to  $M(A_0)$ , then  $M(B'_0)$  cannot be homeomorphic to  $M$  because  $M(A)$  is not homeomorphic to  $I^2$ . Therefore,  $M(B_0)$  is homeomorphic to  $M(B'_0)$  and  $M(A_0)$  is homeomorphic to  $M(A'_0)$ .

If there exists only one surface  $M(A_0)$ , then there exist more than one arcs  $J_i \subset \partial M(A_0)$  such that they are components of preimage of a component of the set  $nX \cap \bar{A}_0$  by  $g_{A_0}$ . We construct a new surface  $M$  gluing two arcs and we construct an analogous surface  $M'$ . We use similar consideration as before and we obtain that also  $M(A_0)$  is homeomorphic to  $M(A'_0)$ .

If  $M(B_0)$  is homeomorphic to  $S^1 \times I$  but  $nY \cap \bar{B}_0 \neq \emptyset$ , we have an arc  $J \subset \partial M(B_0)$  such that  $J$  is a component of a preimage of a component of  $g_{B_0}^{-1}(nY \cap \bar{B}_0)$ . The homeomorphism  $F_{A_0, B_0}$  maps  $M(A_0) \times J$  onto  $M(A'_0) \times J'$ , where an arc  $J' \subset \partial M(B'_0)$  such that  $J'$  is a component of a preimage of a component of  $g_{B'_0}^{-1}(nY' \cap \bar{B}'_0)$ . We can glue two copies  $M(B_0)$  by identification of the arcs  $J$  and glue two copies of  $M(B'_0)$  by identification of the arcs  $J'$ . We obtain new surfaces  $M$  and  $M'$  different from  $S^1 \times I$  and homeomorphism between  $M(A_0) \times M$  and  $M(A'_0) \times M'$ . So  $M(A_0)$  and  $M(A'_0)$  are homeomorphic.

Similarly, if  $M(B_0)$  is homeomorphic to a disk  $I^2$  but the boundary of  $M(B_0)$  contains more than two arcs  $J_i$  that are components of preimages of  $nY \cap \bar{B}_0$  by  $g_{B_0}$ , and analogously  $J'_i$  for  $M(B'_0)$ , the homeomorphism  $F_{A_0, B_0}$  maps the sets  $M(A_0) \times J_i$  onto  $M(A'_0) \times J'_i$ . We can construct new surfaces  $M, M'$  by identification arcs  $J_1$  and  $J_2$  in two copies of  $M(B_0)$  and arcs  $J'_1$  and  $J'_2$  in two copies of  $M(B'_0)$  and a homeomorphism between  $M(A_0) \times M$  and  $M(A'_0) \times M'$  mapping  $M(A_0) \times J_i$  onto  $M(A'_0) \times J'_i$  for one of the remaining arcs. We obtain the previous situation and the surfaces  $M(A_0)$  and  $M(A'_0)$  are homeomorphic.

In the remaining case all the surfaces  $M(B)$  are homeomorphic to disks and their boundaries contain not more than two arcs as before. Two cases are possible. The polyhedron  $Y$  is a Cartesian product of an interval and a graph or  $Y$  contains a Möbius



band  $M$ , which is a union of surfaces  $M(B)$ . Then we can construct a homeomorphism between  $M(A) \times M$  and  $M(A') \times M$ , hence again  $M(A)$  and  $M(A')$  are homeomorphic.

**Remark 4.2.** If we assume in the above Lemma that not  $nX \neq \emptyset$  but  $nY \neq \emptyset$ , then also  $M(A) \approx M(A')$ . If both  $nX$  and  $nY$  are empty, then  $X$  and  $Y$  are surfaces and it is possible that  $X \approx Y'$  and  $Y \approx X'$ .

Hence, if  $Y$  and  $Y'$  are not homeomorphic to products of intervals and graphs, then  $nX$  and  $nX'$  are homeomorphic and all surfaces  $M(A)$  and  $M(A')$  are homeomorphic. We need to prove that the gluings of the surfaces to non-Euclidean parts are the same in both cases. We present the next:

**Lemma 4.2.** *Let  $F: X \times Y \rightarrow X' \times Y'$  be a homeomorphism such that  $F(X \times nY) = X' \times nY'$  and  $Y$  is not homeomorphic to a bundle of intervals over a graph. If  $X$  and  $X'$  are not homeomorphic prime 2-polyhedra, then there exists such a surface  $N$  not homeomorphic to a disk, an annulus or Möbius band such that  $X \times N$  is homeomorphic to  $X' \times N$ .*

(Unless the case where  $X, Y, X', Y'$  are surfaces and  $X \approx Y'$  and  $Y \approx X'$ .)

After proving of this Lemma we need only to indicate that if such 2-polyhedra do not exist.

**Proof.** If the polyhedron  $Y$  has local cut points or  $n_2Y \neq \emptyset$  or  $nY$  contains a simple closed curve or there exists a point  $x \in nY$  such that its regular neighborhood in  $Y$  is not homeomorphic to the set  $\text{cone}\{1, \dots, m\} \times I$ , then  $Y'$  satisfies the same conditions and the polyhedra  $X$  and  $X'$  are homeomorphic. In the opposite case we obtain our polyhedron  $Y$  gluing separate arcs lying in the boundaries of surfaces  $M(B)$ , where  $B \in \square(Y \setminus nY)$ . Let us observe that if  $F(X \times B) = X' \times B'$ , where  $B' \in \square(Y' \setminus nY')$ , then the polyhedra  $X \times M(B)$  and  $X' \times M(B')$  are homeomorphic.

Because  $F(X \times nY) = X' \times nY'$ , we can define a homeomorphism  $F_B: X \times M(B) \rightarrow X' \times M(B')$  by the formula

$$F_B(x, [\{y_n\}]) = (P_1(\lim F(x, y_n), [\{P_2 F(x, y_n)\}]),$$

where  $P_1, P_2$  are the projections on the first and the second factor. The homeomorphism  $F$  maps a regular neighborhood of  $X \times nY$  on a regular neighborhood of  $X' \times nY'$ , so our formula is correct. (We can obtain the same result removing from  $X \times \bar{B}$  and  $X' \times \bar{B}'$  regular neighborhoods of  $X \times nY$  and  $X' \times nY'$ .)

If  $nX \neq \emptyset$ , then  $M(B) \approx M(B')$  by Lemma 4.1. If  $nY \neq \emptyset$ , then also  $M(B) \approx M(B')$  by Remark 4.2. If one of surfaces  $M(B)$  is different from  $I^2$  or  $I \times S^1$  we have our manifold  $N = M(B)$ . The Cartesian products  $X \times N$  and  $X' \times N$  are homeomorphic.

If all surfaces  $M(B)$  are homeomorphic to  $I^2$  or  $I \times S^1$  we consider the arcs  $J_i$ , which are the components of  $g_B^{-1}(nY)$  and analogous arcs  $J'_i$  in  $M(B')$ , where  $g_B$  is defined as in Property 4.2. We have  $F_B(X \times J_i) = X' \times J'_i$ . Because  $Y$  is not a bundle of intervals over a graph, there exists a surface  $M(B)$  homeomorphic to  $S^1 \times I$  with an arc  $J_i$  in its boundary or a surface  $M(B)$  homeomorphic to a disk with at least tree arcs  $J_i$  in its boundary. So we construct the surface  $M$  as in the proof of the previous lemma.

**Lemma 4.3.** *If  $X$  and  $X'$  are 2-polyhedra and  $N$  is an orientable surface not homeomorphic to  $I^2$  or  $I \times S^1$  and the Cartesian products  $X \times N$  and  $X' \times N$  are homeomorphic then the polyhedra  $X$  and  $X'$  are homeomorphic.*

**Proof.** If  $X$  is a surface, then our Lemma is a special case of Theorem 1.1. In the opposite case  $nX \neq \emptyset$  and a homeomorphism  $F: X \times N \rightarrow X' \times N$  maps the set  $nX \times N$  onto  $nX' \times N$ . The sets  $nX$  and  $nX'$  are homeomorphic and all surfaces  $M(A)$  and  $M(A')$ , where  $A \in \square(X \setminus nX)$ ,  $A' \in \square(X' \setminus nX')$  and  $F(A \times N) = A' \times N$ , are homeomorphic by Lemma 4.1. By Property 4.3, there exists a homeomorphism  $F_A: M(A) \times N \rightarrow M(A') \times N$  such that  $(g_{A'} \times \text{id})F_A = F|_{\bar{A} \times N}(g_A \times \text{id})$ . We will compare the gluing maps  $g_A: M(A) \rightarrow \bar{A}$  and  $g_{A'}: M(A') \rightarrow \bar{A}'$ . More precisely, we should find such homeomorphisms  $f: nX \rightarrow nX'$  and  $f_A: M(A) \rightarrow M(A')$ , for all  $A \in \square(X \setminus nX)$  that for each  $x \in g_A^{-1}(nX)$  the formula  $f g_A(x) = g_{A'} f_A(x)$  is true. Then we can define a homeomorphism  $\tilde{f}: X \rightarrow X'$  by the formula

$$\begin{aligned} \tilde{f}(x) &= f(x) \quad \text{for } x \in nX, \\ \tilde{f}(x) &= g_{A'} f_A(g_A^{-1}(x)) \quad \text{for } x \in \bar{A}. \end{aligned}$$

We can require from our homeomorphism  $f: nX \rightarrow nX'$  that if for a subpolyhedron  $K \subset nX$  we have  $F(K \times N) = K' \times N$ , then  $f(K) = K'$ . In particular, we have the conditions:

- (i)  $f(nX \cap \bar{A}) = nX' \cap \bar{A}'$  for  $A \in \square(X \setminus nX)$ ;
- (ii)  $f$  maps the 1-dimensional part of  $X$  on the 1-dimensional part of  $X'$ ;
- (iii)  $f(n_2 X) = n_2 X'$ ;
- (iv)  $f$  maps local cut points of  $X$  on local cut points of  $X'$ ;
- (v)  $f$  maps the set of points of  $nX$  such that they have a regular neighborhood of the form  $\text{cone}\{1, \dots, m\} \times I$  in  $\bar{A}$  on the set of points of  $nX'$  such that they have a regular neighborhood of the form  $\text{cone}\{1, \dots, m\} \times I$  in  $\bar{A}'$ .

It is obvious that the set  $g_A^{-1}(nX \cap \bar{A})$  consists of single points lying in the interior of  $M(A)$  and single points, arcs and simple closed curves lying in the boundary of  $M(A)$ . There exists such a triangulation of  $M(A)$  that the function  $g_A$  cut to each simplex is a homeomorphism.

Let us consider the single points in the interior of  $M(A)$ . Let  $x \in g_A^{-1}(nX \cap \bar{A}) \cap \text{int } M(A)$ . If  $y = g_A(x) \in nX$ , then  $F(y \times N) = y' \times N$ ,  $f(y) = y' \in nX'$  and there exists  $x' \in \text{int } M(A')$  such that  $y' = g_{A'}(x')$ . Because interiors of surfaces are uncoherent and the sets  $g_A^{-1} \cap \text{int } M(A)$  and  $g_{A'}^{-1} \cap \text{int } M(A')$  consist of the same number of points, the position of the points  $x$  and  $x'$  in  $M(A)$  and  $M(A')$  is inessential. We can require that  $f_A(x) = x'$ .

Single points of  $g_A^{-1}(nX \cap \bar{A})$  lying in  $\partial M(A)$  have adequate points in  $\partial M(A')$ , but we need to make sure that if two such points belong to one component of  $\partial M(A)$ , then the adequate points belong to the same component of  $\partial M(A')$  and the order of these points in the adequate components of the boundaries of both surfaces is the same.

If a simple closed curve  $S_i$  is contained in  $g_A^{-1}(nX \cap \bar{A})$ , then  $S_i$  is a component of  $\partial M(A)$  and  $F_A(S_i \times N) = S'_i \times N$ , where  $S'_i$  is a component of  $\partial M(A')$ .

If  $g_A(S_i)$  contains distinguished points  $y$  (that is isolated local cut points, points of  $n_2X$  or such points that their regular neighborhood in  $X$  is not homeomorphic to  $\text{cone}\{1, \dots, n\} \times I$ ), then  $F(\{y\} \times N) = \{y'\} \times N$  where  $y'$  is an analogous distinguished point of  $X'$ . Therefore,  $f(y) = y'$  for each such point. Also,  $F_A(\{x\} \times N) = \{x'\} \times N$ , where  $g_A(x) = y$  and  $g_{A'}(x') = y'$ . We can assume that  $f_A(x) = x'$  and  $f_A$  maps arcs of  $S_i$  between points  $x$  on arcs of  $S'_i$  between their images. If both ends of an arc are the same we need to know that  $g_A$  and  $g_{A'}$  map the arcs on the loops with the same orientations. But it is true, because  $g_A \times \text{id}$  and  $g_{A'} \times \text{id}$  glue  $M(A) \times N$  and  $M(A') \times N$  to  $nX \times N$  and  $nX' \times N$ .

If  $g_A(S_i)$  does not contain distinguished points and  $S_i$  is mapped onto a loop in  $nX$  then  $\deg g_A|_{S_i} \times \text{id}_N = \deg g'_A|_{S'_i} \times \text{id}_N$ , so  $\deg g_A|_{S_i} = \deg g'_A|_{S'_i}$ .

Therefore, the maps  $g_A|_{S_i}$  and  $g'_A|_{S'_i}$  are the same with respect to an isotopy.

If the surfaces  $M(A)$  and  $M(A')$  are orientable, then we can assume that all components  $S_i$  of  $\partial M(A)$  have orientations inducted from  $M(A)$ . The same is true for  $S'_i$ .

We can use similar arguments to show that the maps  $g_A$  and  $g_{A'}$  map components  $J_i$  of  $g_A^{-1}(nX \cap \bar{A})$  and  $J'_i$  of  $g_{A'}^{-1}(nX' \cap \bar{A}')$  being arcs in the same way in  $nX$  and  $nX'$ . More precisely, we can require that the homeomorphism  $f_A$  has a property  $f_A|_{J_i} = g_{A'}|_{J'_i} f_A|_{J_i}$ .

There is only one difference between glueing of  $S_i$  and  $J_i$ . The maps  $g_A$  and  $g_{A'}$  do not map end-points of arcs  $J_i$  and  $J'_i$  on distinguished points of  $X$  and  $X'$ . But we will show at the end of this lemma that we can assume that if  $x$  is an end-point of  $J_i$ , then  $F_A(\{x\} \times N) = \{x'\} \times N$ , where  $x'$  is an end-point of  $J'_i$ .

The problem is if they lie “in the same way” in  $\partial M(A)$  and  $\partial M(A')$ . More precisely: if adequate components of  $g_A^{-1}(nX \cap \bar{A})$  lie in the same component of  $\partial M(A)$ , then adequate components of  $g_{A'}^{-1}(nX' \cap \bar{A}')$  lie in the same component of  $\partial M(A')$ , if they are in the same order on  $S_i$  and  $S'_i$ , and if the surface  $M(A)$  is orientable we must pay attention to orientation of  $S_i$  and  $S'_i$ . If  $N$  is a disk, an annulus or a Möbius band, then it is not true (see examples before Theorem 1.3) that the above conditions hold. Our manifold  $N$  is different from these surfaces. If  $\partial N = \emptyset$ , the solution is obvious because  $\partial(M(A) \times N) = \partial M(A) \times N$ .

If  $\partial N \neq \emptyset$ , we will use the relative version of Splitting Theorem to prove these conditions.

We know that  $F_A(g_A^{-1}(nX \cap \bar{A}) \times N) = g_{A'}^{-1}(nX' \cap \bar{A}') \times N$ . If an arc  $J_i$  is a component of  $g_A^{-1}(nX \cap \bar{A})$ , then there exists an arc  $J'_i$  which is a component of  $g_{A'}^{-1}(nX' \cap \bar{A}')$  such that  $F_A(J_i \times N) = J'_i \times N$ . Similarly, if a point  $\{x_j\}$  is a component of  $g_A^{-1}(nX \cap \bar{A})$  then there exists a point  $\{x'_j\}$  which is a component of  $g_{A'}^{-1}(nX' \cap \bar{A}')$  such that  $F_A(\{x_j\} \times N) = \{x'_j\} \times N$ . The homeomorphism  $F_A$  maps a regular neighborhood of  $x_j \times N$  in  $\partial(M(A) \times N)$  onto a regular neighborhood of  $\{x'_j\} \times N$  in  $\partial(M(A') \times N)$ . So we can assume that there exist next arcs  $J_j$ , being neighborhoods of  $\{x_j\}$  in  $\partial M(A)$  pairwise disjoint and disjoint with arcs  $J_i$  such that  $F_A(J_j \times N) = J'_j \times N$ , for adequate arcs  $J'_j$ . We need to show that arcs  $J_i$  and  $J_j$  lie in  $\partial M(A)$  “in the same way” as arcs  $J'_i$  and  $J'_j$  lie in  $\partial M(A')$ .

Because  $F_A(\bigcup_{i,j} J \times N) = \bigcup_{i,j} J' \times N$ , we obtain that  $F_A(\partial(M(A) \times N) \setminus \text{int}(\bigcup_{i,j} J \times N)) = \partial(M(A') \times N) \setminus \text{int}(\bigcup_{i,j} J' \times N)$ . So we have a homeomorphism of 3-manifolds with the boundaries.

If our surface  $M(A)$  is orientable, we denote  $M = \partial(M(A) \times N) \setminus \text{int}(\bigcup_{i,j} J \times N)$ . We use the Splitting Theorem to investigate 3-manifold  $M$ . Let  $W = (\partial M(A) \setminus \text{int}(\bigcup_{i,j} J)) \times \partial N$ . The components of  $W$  are annuli and tori, and because  $N \neq S^1 \times I$ , none of them is boundary parallel in  $M$ . Each component of  $(\sigma_W(M), \sigma_{\partial W}(\partial M))$  is homeomorphic to  $(I \times N, \partial I \times N)$ ,  $(S^1 \times N, \emptyset)$  or  $(M(A) \times S^1, \bigcup J \times S^1)$ , so they are Seifert pairs (a I-pair is a Seifert pair also).

The manifold  $M$  is irreducible and boundary-irreducible. For somebody who is familiar with 3-manifold, the proof of the above fact is a simple exercise, but in [10, proof of Theorem 2.1] we outlined it in the case  $\bigcup J = \emptyset$ . A proof at the relative case is similar. A surface  $W$  is incompressible because  $N \neq I^2$ . Since  $W \neq \emptyset$ ,  $M$  is sufficiently-large. Of course,  $W$  is two-sided and properly embedded in  $M$ .

We need to show that  $W$  is minimal.

First, we assume that  $V = W \setminus (S_1 \times S_2)$ , where  $S_1 \times S_2$  is a torus, which is a component of  $W$ , also gives a splitting in the sense of Theorem 1.2. According to  $V$ , we have  $(U, U_0) = ((M(A) \times S_2) \cup (S_1 \times N), \bigcup J \times S_2)$ , as a component of  $(\sigma_W(M), \sigma_{\partial W}(\partial M))$ . The pair  $(U, U_0)$  is not a simple pair (see [6, p. 154]) because the incompressible torus  $S_1 \times S_2$  is not  $\bigcup J \times S_2$  parallel in  $M$  and it is not parallel to a component of  $\partial M \setminus (\bigcup J \times S_2)$ . It is not a I-pair, either. If  $M(A)$  is not a disk or an annulus, then  $U$  is not a Seifert manifold (like in [10]). If  $M(A)$  is a disk or annulus, then  $U$  is a Seifert manifold with a fiber given by  $S_1$ . Hence, the set  $U_0 = \bigcup J \times S_2$  is not saturated. Therefore,  $(U, U_0)$  is not a Seifert pair.

Now we assume that  $V = W \setminus (I_0 \times S_2)$ , where  $I_0 \times S_2$  is an annulus which is a component of  $W$ , also gives a splitting in the sense of Theorem 1.2. According to  $V$ , we have  $(U, U_0) = ((M(A) \times S_2) \cup (I_0 \times N), \bigcup J \times S_2 \cup \partial I_0 \times N)$  as a component of  $(\sigma_W(M), \sigma_{\partial W}(\partial M))$ . The pair  $(U, U_0)$  is not a simple pair because an annulus  $I_0 \times S_2$  does not satisfy the conditions required in the definition. It is not a Seifert pair. The fundamental group of  $U$  is infinite with the trivial center, so the manifold is not a Seifert manifold (see [5, Corollary 8.3] or [7, VI.11.a]). The pair  $(U, U_0)$  is not a I-pair. If  $M(A)$  is not a disk, then  $U$  is not a I-boundle. If  $M(A)$  is a disk then  $U = I^2 \times S_2 \cup I_0 \times N$  is a I-boundle. If we have only two arcs  $J$  on the boundary of a disk, their order is not essential. We consider the case with more than two arcs  $J$ . Then  $U_0 = \bigcup J \times S_2 \cup \partial I_0 \times N$  has more than two components. Therefore  $(U, U_0)$  is not a I-pair. The surface  $W$  is minimal.

So the assumptions of Splitting Theorem hold and decomposition is unique up to ambient isotopy. So  $F_A(M)$  has the same decomposition. Hence, the arcs  $J$  and  $J'$  lie in adequate components of  $\partial M(A)$  and  $\partial M(A')$  and in every component of  $\partial M(A)$  and  $\partial M(A')$  order of arcs  $J$  and adequate arcs  $J'$  is the same. Therefore, the thesis of our lemma holds.

If  $M(A)$  is non-orientable, we consider the double oriented cover of  $M(A) \times N$ . It is homeomorphic to  $\widetilde{M(A)} \times N$  because  $N$  is orientable. Now, we consider a homeomorphism  $\widetilde{F_A} : \widetilde{M(A)} \times N \rightarrow \widetilde{M(A')} \times N$  and repeat the previous consideration.

Let us observe that if  $S_i$  is a component of  $\partial M(A)$  then  $F_A(S_i \times N) = S'_i \times N$ , where  $S'_i$  is a component of  $\partial M(A')$ . Also  $F_A(J_i \times N) = J'_i \times N$  and  $F_A(I_i \times N) = I'_i \times N$ , where  $I_i$  are components of  $S_i \setminus \bigcup J_i$  and  $I'_i$  are components of  $S'_i \setminus \bigcup J'_i$ . Therefore if  $x$  is an end-point of  $J_i$  then  $F_A(\{x\} \times N) = x' \times N$ , where  $x'$  is an end-point of  $J'_i$ . We promised before to prove this detail.

Therefore,  $F(\{g_A(x)\} \times N) = \{g_{A'}(x')\} \times N$  and  $f(g_A(x)) = g_{A'}(x')$ . We can assume that  $f_A(x) = x'$ . So we glue arcs  $J_i$  and  $J'_i$  with the same orientation. The orientations of  $J_i$  and  $J'_i$  are induced from  $S_i$  and  $S'_i$ . If  $M(A)$  and  $M(A')$  are orientable, then they induce orientation of  $S_i$  and  $S'_i$ .

**Lemma 4.4.** *If  $X$  and  $X'$  are 2-polyhedra and  $N$  is a non-orientable surface not homeomorphic to Möbius band and the Cartesian products  $X \times N$  and  $X' \times N$  are homeomorphic, then the polyhedra  $X$  and  $X'$  are homeomorphic.*

**Proof.** The first part of the proof of this Lemma is the same as the proof of the previous lemma. We need to check that if arcs  $J \subset \partial M(A)$  and arcs  $J' \subset \partial M(A')$  have the property

$$F_A(J \times N) = J' \times N,$$

then the arcs  $J$  lie in  $\partial M(A)$  “in the same way” as the arcs  $J'$  in  $\partial M(A')$ . (In the sense of the proof of the previous lemma.)

If a surface  $M(A)$  is orientable we do not have a problem. If  $\tilde{N}$  is a double oriented cover of  $N$  then the manifold  $M(A) \times \tilde{N}$  is the double oriented cover of the manifold  $M(A) \times N$  and the manifold  $M(A') \times \tilde{N}$  is the double oriented cover of the manifold  $M(A') \times N$ . The double oriented cover is unique and we can consider a homeomorphism  $\tilde{F}_A : M(A) \times \tilde{N} \rightarrow M(A') \times \tilde{N}$ . Then we use Splitting Theorem as in the previous lemma. Our splitting is unique because  $N$  is not homeomorphic to the Möbius band, so  $\tilde{N}$  is not homeomorphic to the disk or to the annulus.

If both surfaces are non-orientable, we first construct the double oriented covers of  $\partial(M(A) \times N)$  and  $\partial(M(A') \times N)$ . The construction is the same as in [10].

We consider the manifolds  $M(A) \times S_i$ , where  $S_i$  are components of  $\partial N$ , and  $S_j \times N$  where  $S_j$  are components of  $\partial M(A)$ .

Next, we take the oriented double covers  $\tilde{M}(A)$  and  $\tilde{N}$  of  $M(A)$  and  $N$ . The manifolds  $\tilde{M}(A) \times S_i$  and  $S_j \times \tilde{N}$  are the oriented double covers of  $M(A) \times S_i$  and  $S_j \times N$ . Each of the tori  $S_j \times S_i$  is covered by tori  $S'_j \times S_i$  and  $S''_j \times S_i$  in  $\tilde{M}(A) \times S_i$  and is covered by tori  $S_j \times S'_i$  and  $S_j \times S''_i$  in  $S_j \times \tilde{N}$ .

By identifying  $S'_j \times S_i$  with  $S_j \times S'_i$  and  $S''_j \times S_i$  with  $S_j \times S''_i$ , we obtain the oriented double cover  $\tilde{M}$  of  $\partial(M(A) \times N)$ . It is not essential which circles we denoted by  $S'_i, S'_j$  and  $S''_i, S''_j$  because in every case we obtain the unique oriented double cover of  $\partial(M(A) \times N)$ .

Analogously, we construct the oriented double cover  $\tilde{M}'$  of  $\partial(M(A') \times N)$ . Of course,  $\tilde{M}$  and  $\tilde{M}'$  are homeomorphic.

Each of the arcs  $J$  being a component of  $g_A^{-1}(nX \cap \bar{A})$  or regular neighborhood in  $\partial M(A)$  of an isolated point of  $g_A^{-1}(nX \cap \bar{A})$  is covered by two arcs  $J_1, J_2$  and each

of analogous arcs  $J'$  in  $\partial M(A')$  is covered by two arcs  $J'_1, J'_2$ . We can require that  $\tilde{F}(J_i \times \tilde{N}) = J'_i \times \tilde{N}$  because  $F_A(J \times N) = J' \times N$ .

We define 3-manifolds  $M = \tilde{M} \setminus \text{int}(\bigcup J_p \times \tilde{N})$  and  $M' = \tilde{M}' \setminus \text{int}(\bigcup J'_p \times \tilde{N})$ . Each torus  $S_i \times S_j$  has two covering tori  $S_{i,1} \times S_{j,1}$  and  $S_{i,2} \times S_{j,2}$ . We define a surface  $W = \bigcup (S_{i,p} \times S_{j,p}) \setminus \text{int}(\bigcup (J_p \times S_{j,p}))$ . Analogously, for  $\tilde{M}'$  we define  $M'$  and  $W'$ . The manifolds  $M$  and  $M'$  are homeomorphic, so by the Splitting Theorem the splitting of them by the surfaces  $W$  and  $W'$  is unique up to ambient isotopy. This completes the proof of the lemma.

**Proof of Theorem 1.3.** In Lemmas 4.3, 4.4 we proved that if one of the factors of our Cartesian product is a surface not homeomorphic to the disk, annulus or Möbius band, then the decomposition is unique.

If a decomposition into Cartesian product is not unique and no factor is a disk, an annulus or a Möbius band, then one of the factors is a union of the disks glued by arcs contained in their boundaries. No disk has more than two such arcs. So this factor is homeomorphic to a bundle of intervals over a graph.

We prove that one of the factors of the second Cartesian product is homeomorphic to this bundle. We have a homeomorphism  $F: X \times Y \rightarrow X' \times Y'$ . We can assume  $F(X \times nY) = X' \times nY'$ . Then there is one to one correspondence between  $M(A)$  and  $M(A')$ ,  $A \in \square(X \setminus nX)$ ,  $A' \in \square(X' \setminus nX')$  which are homeomorphic, and between disks  $M(B)$  and  $M(B')$ ,  $B \in \square(Y \setminus nY)$ ,  $B' \in \square(Y' \setminus nY')$ . If two arcs from  $\partial M(B)$  are glued, then also two arcs from  $\partial M(B')$  are glued. If two arcs from two boundaries of two disks  $\partial M(B_1)$  and  $\partial M(B_2)$  are glued then two arcs from boundaries of adequate disks  $\partial M(B'_1)$  and  $\partial M(B'_2)$  are glued. If a union of some disks  $M(B)$  is a Möbius band  $N$  and  $N'$  is the union of adequate disks  $M(B')$  then  $M(A) \times N \approx M(A') \times N'$ . Hence, as we proved in [10], the surface  $N'$  is a Möbius band too. If  $N$  is an annulus, then  $N'$  is an annulus, too. Therefore, the bundles  $Y$  and  $Y'$  are homeomorphic.

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